ISRAEL JOURNAL OF MATHEMATICS 154 (2006), 141–155

CHAINS OF WELL-GENERATED BOOLEAN ALGEBRAS WHOSE UNION IS NOT WELL-GENERATED

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ABSTRACT

A Boolean algebra B that has a well-founded sublattice which generates B is called a **well-generated** Boolean algebra. Every well-generated Boolean algebra is superatomic. However, there are superatomic algebras which are not well-generated. We consider two types of increasing chains of Boolean algebras, canonical chains and rank preserving chains, and show that the class of well-generated Boolean algebras is not closed under union of such chains, even when these chains are taken to be countable. A Boolean algebra is **superatomic** iff its Stone space is scattered. If B is superatomic and $a \in B$, then the **rank** of a is the Cantor Bendixon rank of the Stone space of $\{b \mid b \leq a\}$. A chain $\{B_\alpha \mid \alpha < \delta\}$ is a **canonical chain** if for every $\alpha < \beta < \delta$, B_α is the subagebra of B_β generated by all members of B_β whose rank is $< \alpha$. For a superatomic algebra B, $I(B)$ denotes the ideal consisting of all members of B whose rank is less than the rank of B. A chain $\{B_\alpha \mid \alpha < \delta\}$ is a **rank preserving chain** if for every $\alpha < \beta < \delta$ and $a \in I(B_\alpha)$, the rank and mutiplicity of a in B_α are equal to the rank and mutiplicity of a in B_β.

* Supported by The Center For Advanced Studies in Mathematics "Ben Gurion University".
Received September 30, 2004

1. Introduction

A Boolean algebra B is said to be **well-generated** if B has a sublattice G such that G generates B and G is a well-founded subposet of B. Every well-generated Boolean algebra is superatomic, that is, its Stone space is scattered ([BR1] Proposition 2.7(b)). However, there are superatomic algebras which are not well-generated ([BR1] Theorem 3.4(a)).

We consider two types of increasing chains of Boolean algebras, and show that the class of well-generated Boolean algebras is not closed under union of such chains. The class of superatomic Boolean algebras, on the other hand, is closed under these limits, and this last fact is trivial.

Notations: Let B be a Boolean algebra (BA). Then $\mathrm{At}(B)$ denotes the set of atoms of B, and $I^{\mathrm{At}}(B)$ denotes the ideal of B generated by $\mathrm{At}(B)$. Note that $I^{\mathrm{At}}(B)$ may be equal to B. We define by induction on ordinals the sequence of **canonical ideals** of B. Let $I_0(B) = \{0^B\}$. Suppose that the ideal $I_\alpha(B)$ has been defined. Let $\varphi_\alpha \colon B \to B/I_\alpha(B)$ denote the canonical homomorphism from B onto $B/I_\alpha(B)$. Define

$$I_{\alpha+1}(B) = \varphi_\alpha^{-1}(I^{\mathrm{At}}(B/I_\alpha(B))).$$

For a limit ordinal δ define

$$I_\delta(B) = \bigcup_{\gamma<\delta} I_\gamma(B).$$

A Boolean algebra B is **superatomic**, if for some ordinal α, $B/I_\alpha(B)$ is finite. Suppose that $B \neq \{0^B\}$. The first α for which $B/I_\alpha(B)$ is finite is called the **rank** of B, and is denoted by $\mathrm{rk}(B)$. If $B = \{0^B\}$, then $\mathrm{rk}(B)$ is defined to be -1. In the rest of this work we assume that B denotes a superatomic Boolean algebra.

For $a \in B$ we let $B \restriction a$ denote the Boolean algebra induced by B on the set $\{b \in B \mid b \leq a\}$. The **rank** of a in B is defined by $\mathrm{rk}^B(a) = \mathrm{rk}(B \restriction a)$. Note that $I_\alpha(B) = \{a \in B \mid \mathrm{rk}^B(a) < \alpha\}$. Define $I(B) = I_{\mathrm{rk}(B)}(B)$. The multiplicity of $a \in B$ is defined by $\mathrm{mlt}^B(a) = |(B \restriction a)/I((B \restriction a))|$. The algebra B is **unitary**, if $B/I(B) \cong \{0,1\}$.

Definition 1.1: Let B and C be superatomic BA's such that $B \subseteq C$.

(a) $B \subseteq^{\mathrm{rk}} C$, if for every $b \in I(B)$, $\mathrm{rk}^B(b) = \mathrm{rk}^C(b)$ and $\mathrm{mlt}^B(b) = \mathrm{mlt}^C(b)$.

(b) $B \subseteq^{\mathrm{can}} C$, if for some ordinal β, $I(B) = I_\beta(C)$. Note that if $B \subseteq^{\mathrm{can}} C$ and $B \neq C$, then B is unitary.

(c) Let $\vec{B} = \{B_i : i < \alpha\}$ be a chain of superatomic BA's. $\vec{B}$ is a **rank preserving chain**, if for every $i < j < \alpha$, $B_i \subseteq^{\text{rk}} B_j$. $\vec{B}$ is a **canonical chain**, if for every $i < j < \alpha$, $B_i \subseteq^{\text{can}} B_j$.

The next proposition contains facts that follow trivially from the definitions.

PROPOSITION 1.2:

(a) *If $B \subseteq^{\text{can}} C$, then $B \subseteq^{\text{rk}} C$.*

(b) *Let $\vec{B} = \{B_i | \; i < \alpha\}$ be a rank preserving chain and $B = \bigcup_{i<\alpha} B_i$. Then for every $i < \alpha$, $B_i \subseteq^{\text{rk}} B$.*

(c) *Let $\vec{B} = \{B_i | \; i < \alpha\}$ be a canonical chain and $B = \bigcup_{i<\alpha} B_i$. Then for every $i < \alpha$, $B_i \subseteq^{\text{can}} B$.*

The union of a canonical chain of well-generated Boolean algebras need not be well-generated. Indeed any thin tall non-well-generated Boolean algebra B is an example of such an algebra. (Thin tall Boolean algebras are defined in 3.1). Here is an explanation. Let B_α be the subalgebra of B generated by $I_\alpha(B)$. Then $\{B_\alpha | \; \alpha < \aleph_1\}$ is a canonical chain whose union is B. Every B_α is a countable superatomic algebra, and as such, it is well-generated. But the union of the B_α's is not. The existence of non-well-generated thin tall Boolean algebras is shown in Theorem 3.2.

As was explained above, a non-well-generated thin tall Boolean algebra yields a canonical chain of length $\aleph_1$ of well-generated Boolean algebras whose union is not well-generated. It is less obvious how to construct a canonical chain of length ω of well-generated BA's whose union is not well-generated. In Theorem 1.4 we construct such a chain. More specifically, we construct a canonical chain $\{B_n | \; n \in \omega\}$ such that for every n, $\text{rk}(B_n) = n$ and B_n is well-generated, but $B := \bigcup_{n\in\omega} B_n$ is not well-generated. Clearly, $\text{rk}(B) = \omega$.

Clearly, the union of any strictly increasing infinite canonical chain must have infinite rank. So the construction of Theorem 1.4 yields a counter-example with minimal rank. For rank preserving chains, however, one could also ask for counter-examples with finite rank. We shall construct a rank preserving chain of length ω consisting of well-generated unitary Boolean algebras of rank 3 whose union B is not well-generated. The union of a rank preserving chain of rank-3 unitary Boolean algebras must have rank 3. Since rank-2 Boolean algebras are always well-generated, B is a minimal counter-example — indeed, in two senses: the rank of B is minimal, and the length of the chain which forms B is minimal.

We shall thus prove the following two theorems.

THEOREM 1.3: *There is a rank preserving chain $\{B_i | \; i < \omega\}$ such that:*

(1) *For every* $i \in \omega$, $\mathrm{rk}(B_i) = 3$ *and* B_i *is unitary.*
(2) *For every* $i \in \omega$, B_i *is well-generated.*
(3) $B := \bigcup_{i\in\omega} B_i$ *is not well-generated.*
It follows from (1) *that* $\mathrm{rk}(B) = 3$ *and that* B *is unitary.*

THEOREM 1.4: *There is a canonical chain* $\{B_i \mid i < \omega\}$ *such that:*
(1) *For every* $i \in \omega$, $\mathrm{rk}(B_i) = i$.
(2) *For every* $i \in \omega$, B_i *is well-generated.*
(3) $B := \bigcup_{i\in\omega} B_i$ *is not well-generated.*
It follows from (1) *that* $\mathrm{rk}(B) = \omega$ *and that* B *is unitary.*

The Boolean algebras constructed in the above theorems have cardinality $2^{\aleph_0}$, and their sets of atoms are of cardinality $\aleph_1$. We do not know whether these cardinalities are the minimal possible. So the following question may be asked.

QUESTION 1.5: (a) *Is there a canonical chain* $\{B_i \mid i \in \omega\}$ *such that each* B_i *is a well-generated subalgebra of* $\mathcal{P}(\omega)$, *and* $\bigcup_{i\in\omega} B_i$ *is not well-generated?*

(b) *Is there a rank preserving chain* $\{B_i \mid i \in \omega\}$ *such that each* B_i *is a well-generated subalgebra of* $\mathcal{P}(\omega)$ *of rank* 3, *and* $\bigcup_{i\in\omega} B_i$ *is not well-generated?*

(c) *Are there examples in which the* B_i*'s have cardinality* $\aleph_1$*?*

2. The construction of the chains

We describe a certain set-theoretic construction. It will be used in Theorems 1.3 and 1.4.

Definition 2.1: For a cardinal μ and a set K, write $\mathcal{P}_\mu(K) = \{a \subseteq K \mid |a| = \mu\}$. We shall deal with objects which we call candidates. Let K be an uncountable set. A **candidate** for K is a sequence $\vec{A} = \{A_i \mid i \in \omega\}$ such that for every $i \in \omega$, $A_i \subseteq \mathcal{P}_{\aleph_0}(K)$, for every distinct $i, j \in \omega$, $|A_i| \geq \aleph_0$, $A_i \cap A_j = \emptyset$, and $\bigcup_{i\in\omega} A_i$ is an almost disjoint family. That is, $a \cap b$ is finite for every distinct $a, b \in \bigcup_{i\in\omega} A_i$.

(a) Suppose that $\vec{A}$ is a candidate for K. We call $\vec{A}$ an **intersection system** for K, if for every countable partition P of K there is $p \in P$ such that $\mathrm{IS}(\vec{A}, p)$ is infinite, where $\mathrm{IS}(\vec{A}, p) := \{i \in \omega \mid (\exists a \in A_i)(|a \cap p| = \aleph_0\}$.

(b) Suppose that $\vec{A}$ is a candidate for K and set $A = \bigcup_{i\in\omega} A_i$. Let $v\colon A \to \mathcal{P}_{\aleph_0}(K)$. Call $\langle \vec{A}, v\rangle$ a **spaced intersection system** for K, if the following conditions hold.

(1) For every $a \in A$, $v(a) \subseteq a$, and $v(a)$ and $a - v(a)$ are infinite.
(2) For every countable partition P of K there is $p \in P$ such that $\mathrm{IS}(\vec{A}, v, p)$ is infinite, where $\mathrm{IS}(\vec{A}, v, p) := \{i \in \omega \mid (\exists a \in A_i)(|v(a) \cap p| = \aleph_0)\}$.

(c) Let $\mu \leq |K|$ and $\vec{A}$ be a candidate for K. We say that $\vec{A}$ is a **μ-deep filling system** for K, if for every $b \in \mathcal{P}_\mu(K)$ and $i \in \omega$, $A_i \cap \mathcal{P}(b) \neq \emptyset$. A $|K|$-deep filling system for K is called a **filling system** for K.

Intersection systems will be used in the construction of Theorem 1.3 and spaced intersection systems will be used in the construction of Theorem 1.4. A "filling system" is a stronger notion in the following sense. If $\vec{A}$ is a filling system for K, then it is an intersection system for K, and if v is any function from $\bigcup_i A_i$ satisfying Clause 1 of Part (b), then $\langle \vec{A}, v\rangle$ is a spaced intersection system. These facts are stated in Part (b) of the next proposition. All claims in the next proposition are trivialities, so their proofs are omitted.

PROPOSITION 2.2: (a) *Let $\vec{A} = \{A_i \mid i \in \omega\}$ be a μ-deep filling system for K, and $L \subseteq K$ be such that $|L| \geq \mu$. Then $\{A_i \cap \mathcal{P}(L) \mid i \in \omega\}$ is a μ-deep filling system for L.*

(b) *Let $\vec{A} = \{A_i \mid i \in \omega\}$ be a filling system for K. Then:*

(i) *$\vec{A}$ is an intersection system for K.*

(ii) *Let v: $\bigcup_{i\in\omega} A_i \to \mathcal{P}_{\aleph_0}(K)$ be such that for every $a \in \bigcup_{i\in\omega} A_i$, $v(a) \subseteq a$, and $v(a)$ and $a - v(a)$ are infinite. Then $\langle \vec{A}, v\rangle$ is a spaced intersection system for K.*

(c) *Suppose that $\vec{A}$ is an intersection system for K and $K \subseteq M$. Then $\vec{A}$ is an intersection system for M. The analogous statement for spaced filling systems also holds.*

The next theorem establishes the existence of an $\aleph_1$-deep filling system for $2^{\aleph_0}$. By Proposition 2.2(a), this implies the existence of a filling system for $\aleph_1$, and from2.2(b) and (c) it follows that every $\lambda > \aleph_1$ has an intersection system. This means that the existence of intersection systems is of interest only for $\aleph_1$. The same is of course true also for spaced intersection systems. For filling systems, however, the situation is different, and in Theorem 3.3 we shall show the existence of filling systems for some cardinals $> 2^{\aleph_0}$. However, these additional filling systems have no Boolean algebraic application.

THEOREM 2.3: (a) *$\mathbb{R}$ has an $\aleph_1$-deep filling system.*

(b) *$\aleph_1$ has a filling system.*

Proof: Part (b) follows trivially from (a) and Proposition 2.2(a), so we prove (a). Let $\{e_\alpha \mid \alpha < 2^{\aleph_0}\}$ be an enumeration of all subsets of $\mathbb{R}$ which are order isomorphic to $\mathbb{Q}$. We define by induction on $\alpha < 2^{\aleph_0}$ a sequence $\vec{A}_\alpha = \{A_{\alpha,i} \mid i \in \omega\}$. For every i, $A_{\alpha,i} \subseteq \mathcal{P}_{\aleph_0}(\mathbb{R})$. The induction hypotheses are as follows:

(1) For every $i \in \omega$, $|A_{\alpha,i}| \leq |\alpha| + \aleph_0$.

(2) For every $i \neq j$, $A_{\alpha,i} \cap A_{\alpha,j} = \emptyset$. Also, $\bigcup_{i\in\omega} A_{\alpha,i}$ is an almost disjoint family, and for every $a \in \bigcup_{i\in\omega} A_{\alpha,i}$, a is the range of a convergent sequence in $\mathbb{R}$.

(3) If $\alpha < \beta$, then $A_{\alpha,i} \subseteq A_{\beta,i}$ for every $i \in \omega$.

Let $A_{0,i} = \emptyset$ for every $i \in \omega$. For a limit ordinal δ and $i \in \omega$ let $A_{\delta,i} = \bigcup_{\alpha<\delta} A_{\alpha,i}$.

Suppose that $\vec{A}_\alpha$ has been defined, and we define $\vec{A}_{\alpha+1}$. Let $L_\alpha = \{\lim a |\ a \in \bigcup_{i\in\omega} A_{\alpha,i}\}$. So $|L_\alpha| \leq |\alpha| + \aleph_0$. Since e_α is order isomorphic to $\mathbb{Q}$, the set $\mathrm{Acc}(e_\alpha)$ of its accumulation points is of cardinality $2^{\aleph_0}$. Let $r_\alpha \in \mathrm{Acc}(e_\alpha) - L_\alpha$. Let $\{\vec{a}_{\alpha,i} |\ i \in \omega\}$ be a set of pairwise disjoint sequences converging to r_α such that for every $i \in \omega$, $a_{\alpha,i} := \mathrm{Rng}(\vec{a}_{\alpha,i}) \subseteq e_\alpha$. Let $A_{\alpha+1,i} = A_{\alpha,i} \cup \{a_{\alpha,i}\}$. It is easy to see that $\{A_{\alpha+1,i} |\ i \in \omega\}$ satisfies the induction hypotheses.

For every $i \in \omega$ let $A_i = \bigcup\{A_{\alpha,i} |\ \alpha < 2^{\aleph_0}\}$ and $\vec{A} = \{A_i |\ i \in \omega\}$. We show that $\vec{A}$ is an $\aleph_1$-deep filling system for $\mathbb{R}$. Clearly, $\vec{A}$ is a candidate for $2^{\aleph_0}$. Let $c \subseteq \mathbb{R}$ be of cardinality $\aleph_1$. Then c contains a subset isomorphic to the rationals. Hence for some $\alpha < 2^{\aleph_0}$, $e_\alpha \subseteq c$. Hence for every $i \in \omega$, $a_{\alpha,i} \subseteq e_\alpha \subseteq c$. Since $a_{\alpha,i} \in A_i$, it follows that $A_i \cap \mathcal{P}(c) \neq \emptyset$. So $\vec{A}$ is an $\aleph_1$-deep filling system for $2^{\aleph_0}$. ∎

QUESTION 2.4: *If $\vec{A} = \{A_i |\ i \in \omega\}$ is an intersection system for K, define $\|\vec{A}\| := |\bigcup_{i\in\omega} A_i|$. In view of Theorem 2.3 and Proposition 2.2, is it consistent that $\aleph_1 < 2^{\aleph_0}$ and $\aleph_1$ has an intersection system $\vec{A}$ with $\|\vec{A}\| = \aleph_1$?*

We observe that MA $+\aleph_1 < 2^{\aleph_0}$ implies that such an intersection system does not exist.

PROPOSITION 2.5: *Assume* MA $+\aleph_1 < 2^{\aleph_0}$, *let $\lambda < 2^{\aleph_0}$ and $\{a_i |\ i < \lambda\}$ be an almost disjoint family of countable subsets of $\aleph_1$. Then there is a partition P of $\aleph_1$ into countably many sets such that $p \cap a_i$ is finite for every $p \in P$ and $i < \lambda$.*

Proof: Let $\langle R, \leq\rangle$ be the following poset. Every member of R has the form $\langle \sigma, \eta\rangle$, where

(1) σ is a finite subset of $\omega \times \aleph_1$, and for every $\alpha \in \aleph_1$ there is at most one $n \in \omega$ such that $\langle n, \alpha\rangle \in \sigma$,

(2) η is a finite subset of λ.

For $\langle \sigma_1, \eta_1\rangle, \langle \sigma_2, \eta_2\rangle \in R$ we define $\langle \sigma_1, \eta_1\rangle \leq \langle \sigma_2, \eta_2\rangle$ if $\sigma_1 \subseteq \sigma_2$, $\eta_1 \subseteq \eta_2$ and for every $i \in \eta_1$ and $n \in \mathrm{Dom}(\sigma_1)$, $(\{n\} \times a_i) \cap \sigma_1 = (\{n\} \times a_i) \cap \sigma_2$. The poset

$\langle R, \leq\rangle$ is c.c.c., and the proof of this fact is standard and is left to the reader. For every $\alpha < \aleph_1$ let $D_\alpha = \{\langle\sigma,\eta\rangle \in R \mid \alpha \in \mathrm{Rng}(\sigma)\}$ and for every $i < \lambda$ let $E_i = \{\langle\sigma,\eta\rangle \in R \mid i \in \eta\}$. Then the D_α's and the E_i's are dense in R. Let G be a filter which intersects each D_α and each E_i and let $P = \{p_n \mid n \in \omega\}$, where $p_n = \{\alpha \mid \text{for some } \langle\sigma,\eta\rangle \in G, \langle n,\alpha\rangle \in \sigma\}$. Then P is as required in the proposition. ∎

We prepare for the proof of Theorem 1.3. Suppose that $\vec{A} = \{A_i \mid i \in \omega\}$ is an intersection system for K. We shall assume that $\bigcup\{a \mid a \in \bigcup_i A_i\} = K$. This is really not needed, but it makes the notation somewhat simpler, and in fact, it does not change the generality of the construction.

Definition 2.6: Let κ be an uncountable cardinal and $\vec{A} = \{A_i \mid i \in \omega\}$ be an intersection system for κ such that $\bigcup\{a \mid a \in \bigcup_i A_i\} = \kappa$. We define an increasing sequence of Boolean algebras $B_i(\vec{A})$, $i \in \omega$. The union of this chain is denoted by $B(\vec{A})$. Let B_κ be the subalgebra of $\mathcal{P}(\kappa\times\omega)$ generated by

$$\{\{e\} \mid e \in \kappa\times\omega\} \cup \{\{\alpha\}\times\omega \mid \alpha < \kappa\}.$$

For $i \in \omega$ let $B_i(\vec{A})$ be the subalgebra of $\mathcal{P}(\kappa\times\omega)$ generated by

$$B_\kappa \cup \bigcup_{j\leq i}\{a\times(\omega - \{j\}) \mid a \in A_j\}.$$

Let $B(\vec{A}) = \bigcup_{i\in\omega} B_i(\vec{A})$.

Definition 2.7: Let B be a Boolean algebra. The operations of B are denoted by $+$, $\cdot$ and $-$. The partial ordering of B is denoted by $\leq$, and $\vartriangle$ denotes the symmetric difference, that is, $a \vartriangle b = (a - b) + (b - a)$. Suppose that B is superatomic.

(a) Let $a, b \in B$. Define $a \preceq^B b$, if $a = 0$ or $\mathrm{rk}^B(a - b) < \mathrm{rk}^B(a)$; and $a \sim^B b$, if $a \preceq^B b$ and $b \preceq^B a$. We omit the superscript ^{B}, when B can be understood from the context.

(b) For $\alpha \leq \mathrm{rk}(B)$ let $\widehat{\mathrm{At}}_\alpha(B) = \{a \in B \mid a/I_\alpha(B) \in \mathrm{At}(B/I_\alpha(B))\}$. Let $\widehat{\mathrm{At}}(B) = \bigcup\{\widehat{\mathrm{At}}_\alpha(B) \mid \alpha \leq \mathrm{rk}(B)\}$.

(c) Let $E \subseteq B$. We say that E is a **complete set of representatives (CSR)** for B, if $E \subseteq \widehat{\mathrm{At}}(B)$, and for every $a \in \widehat{\mathrm{At}}(B)$ there is a unique $e \in E$ such that $e \sim^B a$.

(d) A Boolean algebra B is **canonically well-generated**, if it has a CSR H such that the sublattice of B generated by H is well-founded.

It is easy to see that if H is a CSR for B, then H generates B. So a canonically well-generated BA is indeed well-generated. We shall need the following claims which appear in [BR1].

PROPOSITION 2.8: (a) *Let H be a CSR for B. Then for every $a \in B$ there is a finite set $\sigma \subseteq H$ such that for every $b \in \sigma$, $\mathrm{rk}^B(b) \leq \mathrm{rk}^B(a)$ and $a \leq \sum \sigma$.*

(b) (i) *Let H be a CSR for B. Suppose that for every $a, b \in H$: if $a \preceq b$, then $a \leq b$. Then the lattice generated by H is well-founded.*

(ii) *Suppose that B is canonically well-generated. Then B has a CSR H such for every $a, b \in H$: if $a \preceq b$, then $a \leq b$.*

(c) *Let B be a well-generated BA and I be a maximal ideal in B. Then there is a well-founded lattice $G \subseteq I$ such that G generates B.*

(d) *Every countable superatomic Boolean algebra is canonically well-generated.*

Parts (a)–(d) of 2.8 are proved in [BR1]: Proposition 3.7(a), Proposition 2.10, Proposition 2.9(3) and Proposition 3.3(d) respectively.

Proof of Theorem 1.3: We shall show that if $\vec{A} = \{A_i \mid i \in \omega\}$ is an intersection system for κ, and $\kappa = \bigcup_{i\in\omega} \bigcup A_i$, then $\{B_i(\vec{A}) \mid i \in \omega\}$ and $B(\vec{A})$ fulfill the requirements of Theorem 1.3. Since by Theorem 2.3 and Proposition 2.2(b), $\aleph_1$ has an intersection system, Theorem 1.3 follows. Indeed, if $\vec{A}$ is taken to be the intersection system for $\aleph_1$ which was constructed in Theorem 2.3 and Proposition 2.2(b), then $|B(\vec{A})| = 2^{\aleph_0}$ and $|\,\mathrm{At}(B(\vec{A}))| = \aleph_1$.

Let $\vec{A}$ and κ be as above, and denote $B_i = B_i(\vec{A})$ and $B = B(\vec{A})$.

CLAIM: *The following facts hold.*

1. *For every $i \in \omega$, $\mathrm{At}(B_i) = \{\{e\} \mid e \in \kappa \times \omega\}$.*
2. *For every $i \in \omega$, $\widehat{\mathrm{At}}_1(B_i) = \{(\{\alpha\} \times \omega) \triangle \sigma \mid \alpha < \kappa$ and $\sigma \in \mathcal{P}_{<\aleph_0}(\kappa \times \omega)\}$.*
3. *For every $i \in \omega$, $\mathrm{rk}(B_i) = 3$ and B_i is unitary. This implies that $\mathrm{rk}(B) = 3$ and B is unitary.*
4. *For every $i, j \in \omega$: if $i < j$, then $\widehat{\mathrm{At}}_2(B_i) = B_i \cap \widehat{\mathrm{At}}_2(B_j)$.*
5. *For every $i \in \omega$, B_i is canonically well-generated.*
6. *B is not well-generated.*

Proof: It is left to the reader to check that Facts (1)–(4) hold.

We prove (5). For $i \in \omega$, $\alpha \in \kappa$ and $a \subseteq \kappa$ denote

$$c(\alpha, i) = \{\alpha\} \times (\omega - \{0, \dots, i\}) \quad \text{and} \quad d(a, i) = a \times (\omega - \{i\}).$$

Let

$$H_0 = \{\{e\} \mid e \in \kappa\times\omega\}, \quad H_1(i) = \{c(\alpha, i) \mid \alpha \in \kappa\},$$
$$H_2(i) = \bigcup_{j\leq i}\{d(a,j) \mid a \in A_j\} \quad \text{and} \quad H(i) = H_0 \cup H_1(i) \cup H_2(i).$$

It is easy to see that $H(i)\cup\{1^{B_i}\}$ is a CSR for B_i. Observe that for every $a, b \in H(i)$: if $a \preceq b$, then $a \subseteq b$. So by Proposition 2.8(b)(i), the lattice generated by $H(i) \cup \{1^{B_i}\}$ is well-founded. Hence B_i is canonically well-generated.

We prove (6). Suppose by contradiction that G is a well-founded lattice which generates B. Since B is unitary, $I(B)$ is a maximal ideal. By Proposition 2.8(c), we may assume that $G \subseteq I(B)$. Observe that for every $a \in I(B)$ there is $g \in G$ such that $a \leq g$. So $g_a := \min(\{g \in G \mid a \preceq g\})$ is well-defined. For $\alpha \in \kappa$ set $c(\alpha) = \{\alpha\}\times\omega$ and

$$n(\alpha) = \min(\{n \in \omega \mid g_{c(\alpha)} \supseteq \{\alpha\} \times (\omega - \{0,\dots,n-1\})\}).$$

For $\ell \in \omega$ let $K_\ell = \{\alpha \in \kappa \mid n(\alpha) = \ell\}$. Clearly, $\{K_\ell \mid \ell \in \omega\}$ is a partition of κ. So since $\vec{A}$ is an intersection system, there is $n_0 \in \omega$ such that $\mathrm{IS}(\vec{A}, K_{n_0})$ is infinite. Let $k \in \mathrm{IS}(\vec{A}, K_{n_0})$ be such that $k \geq n_0$, and let $a \in A_k$ be such that $a \cap K_{n_0}$ is infinite.

We note that $(*)$ if $d \in I(B)$, then $d \cap (a\times\{k\})$ is finite. To see this, consider the set H defined as follows. $H = H_0 \cup \{c(\alpha) \mid \alpha \in \kappa\} \cup \bigcup_{i\in\omega} H_2(i)$. Then $H \cup \{1^B\}$ is a CSR for B, and for every $d \in H$, $d \cap (a\times\{k\})$ is finite. By Proposition 2.8(a), every member of $I(B)$ is contained in a finite union of members of H. This implies $(*)$.

Recall that $a \in A_k$. So $d{:=}a\times(\omega - \{k\}) \in B$. From $(*)$ and the fact: $g_d \in I(B)$, it follows that $g_d \cap (a\times\{k\})$ is finite. So if we define $a^* = \{\alpha \in a \mid \langle\alpha, k\rangle \notin g_d\}$, then $a - a^*$ is finite. Let $a' = \{\alpha \in a \mid c(\alpha) \preceq g_d\}$. Since $d \preceq g_d$, it follows that $a - a'$ is finite. We thus have

$$a^* \cap a' \subseteq a \quad \text{and} \quad a - (a^* \cap a') \text{ is finite.}$$

Recall that $a \cap K_{n_0}$ is infinite. It thus follows that $a^* \cap a' \cap K_{n_0}$ is infinite. Let $\alpha \in a^* \cap a' \cap K_{n_0}$. Since $\alpha \in a'$, we conclude that $c(\alpha) \preceq g_d$. Since $\alpha \in K_{n_0}$ and $k \geq n_0$, it follows that $\langle\alpha, k\rangle \in g_{c(\alpha)}$, and since $\alpha \in a^*$, we have that $\langle\alpha, k\rangle \notin g_d$. That is, $g_{c(\alpha)} \not\subseteq g_d$. This contradicts the fact that $g_{c(\alpha)}$ is the minimal member of G almost containing $c(\alpha)$. It follows that B is not well-generated. We have proved the claim.

That $B_i \subseteq^{\mathrm{rk}} B_j$, follows easily from Facts (1)–(4), and Fact (5) is stronger than saying that B_i is well-generated. So $\{B_i | \ i \in \omega\}$ is a rank preserving chain of rank-3 well-generated Boolean algebras which, by Fact (6), has a non-well-generated union. ∎

The above example is slightly stronger that what was stated in Theorem 1.3. Namely, each B_i is canonically well-generated and not just well-generated.

The construction of the Boolean algebra of Theorem 1.4

Let H be a subset of a Boolean algebra B. Then $\mathrm{cl}_B^{\mathrm{bln}}(H)$ denotes the Boolean subalgebra of B generated by H and $\mathrm{cl}_B^{\mathrm{rng}}(H)$ denotes the Boolean ring generated by H. That is, $\mathrm{cl}_B^{\mathrm{rng}}(H)$ is the closure of H under join, meet and difference. The subscript $_B$ is usually omitted.

Let I be a Boolean ring without a unit. Then the Boolean algebra generated by I is denoted by $\mathrm{BL}(I)$. That is, $\mathrm{BL}(I)$ is characterized by:

(i) I is a subring of $\mathrm{BL}(I)$,

(ii) I is a maximal ideal in $\mathrm{BL}(I)$.

If $\mathrm{BL}(I)$ is superatomic, then $\mathrm{rk}(I)$ is defined to be $\mathrm{rk}(\mathrm{BL}(I))$. Also denote $\widehat{\mathrm{At}}(I) = \widehat{\mathrm{At}}(\mathrm{BL}(I)) \cap I$. The notions of a well-generated and canonically well-generated rings are defined in the obvious way.

Also, note that if a Boolean ring I is well-generated, then $\mathrm{BL}(I)$ is well-generated, and the same is true for canonical well-generatedness.

Let $\langle \{A_i | \ i \in \omega\}, v\rangle$ be a spaced intersection system for κ such that $\bigcup\{a | \ a \in \bigcup_i A_i\} = \kappa$, and set $\vec{A} = \{A_i | \ i \in \omega\}$ and $A = \bigcup_{i\in\omega} A_i$. We construct the Boolean algebra $B(\vec{A}, v)$.

For $a \in A$ let $n(a)$ denote the number n such that $a \in A_n$. For every $a \in A$ let $B(a)$ be a Boolean algebra with the following properties.

(1) $\{\{\alpha\} | \ \alpha \in a\} \subseteq B(a) \subseteq \mathcal{P}(a)$.

(2) $B(a)$ is countable and unitary.

(3) $\mathrm{rk}(B(a)) = n(a)$.

(4) For every $b \in I(B(a))$, $b \cap v(a)$ is finite.

Denote $I(a) = I(B(a))$. So $I(a)$ is a Boolean subring of $\mathcal{P}(a)$. Let $d(a) = a\times\omega - v(a)\times\{n(a)\}$. We define subrings of $\mathcal{P}(a\times\omega)$.

$$I^0(a) = \{b\times\omega | \ b \in I(a)\},$$
$$I^1(a) = \mathrm{cl}^{\mathrm{rng}}(I^0(a) \cup \{\{e\} | \ e \in a\times\omega\}),$$
$$I^2(a) = \mathrm{cl}^{\mathrm{rng}}(I^1(a) \cup \{d(a)\}).$$

Let $B(\vec{A}, v) = \mathrm{cl}^{\mathrm{bln}}_{\mathcal{P}(\kappa\times\omega)}(\bigcup_{a\in A} I^2(a))$.

Proof of Theorem 1.4: Let κ be an uncountable cardinal, $\langle \vec{A}, v\rangle = \langle \{A_i \mid i \in \omega\}, v\rangle$ be a spaced intersection system for κ such that $\bigcup_{i\in\omega} \bigcup A_i = \kappa$ and $B = B(\vec{A}, v)$. For $i \in \omega$ let B_i be the subalgebra of B generated by $I_i(B)$. It will be shown that B and $\{B_i \mid i \in \omega\}$ fulfill the requirements of Theorem 1.4. By Theorem 2.3 and Proposition 2.2(b), $2^{\aleph_0}$ has a spaced intersection system. So Theorem 1.4 follows.

Three facts have to be shown.

CLAIM 1: $\mathrm{rk}(B) = \omega$ *and* B *is unitary.*

CLAIM 2: *For every* $n \in \omega$, $\mathrm{cl}^{\mathrm{bln}}(I_n(B))$ *is canonically well-generated.*

CLAIM 3: B *is not well-generated.*

Let I be a Boolean ring such that $\mathrm{BL}(I)$ is superatomic and let $H \subseteq I$. We say that H is an **exact set of representatives (ESR)** for I, if:

(1) $H \subseteq \widehat{\mathrm{At}}(I)$.

(2) For every $a \in \widehat{\mathrm{At}}(I)$ there is a unique $b \in H$ such that $b \sim a$.

(3) For every $a, b \in H$: if $a \preceq b$, then $a \leq b$.

The verification of the following facts is left to the reader.

FACT 1: Let $a, a' \in A$ be distinct and $b \in I^2(a')$. Then $b \cap (a\times\omega)$ has the following form: there are finite sets $\sigma \subseteq \omega$ and $\tau \subseteq a\times\omega$ such that $b \cap (a\times\omega) = (\sigma\times\omega)\vartriangle\tau$.

FACT 2: Let $a \in A$, $b \in I^2(a)$ and $c \in B \restriction b$. Then $c \in I^2(a)$. That is, $B \restriction b = I^2(a) \restriction b$.

FACT 3: For every $a \in A$ and $b \in I^2(a)$, $\mathrm{rk}^B(a) = \mathrm{rk}^{I^2(a)}(a)$ and $\mathrm{mlt}^B(a) = \mathrm{mlt}^{I^2(a)}(a)$.

FACT 4: For every $a \in A$ and $e \in I(a)$,

$$\mathrm{rk}^{I^2(a)}(e\times\omega) = 1 + \mathrm{rk}^{I(a)}(e) \quad \text{and} \quad \mathrm{mlt}^{I^2(a)}(e\times\omega) = \mathrm{mlt}^{I(a)}(e).$$

Also,

$$\mathrm{rk}^{I^2(a)}(d(a)) = n(a) + 1 \quad \text{and} \quad \mathrm{mlt}^{I^2(a)}(d(a)) = 1.$$

FACT 5: For every $b \in B - \mathrm{cl}^{\mathrm{rng}}(\bigcup_{a\in A} I^2(a))$, $\mathrm{rk}^B(b) = \omega$ and $\mathrm{mlt}^B(b) = 1$.

FACT 6: $I(B) = \mathrm{cl}^{\mathrm{rng}}(\bigcup_{a\in A} I^2(a))$.

FACT 7: For every $b \in \widehat{\mathrm{At}}(I(B)) - \mathrm{At}(B)$ there are $a \in A$ and $e \in \widehat{\mathrm{At}}(I(a))$ such that $b \sim e \times \omega$ or $b \sim d(a)$.

Proof of Claim 1: The claim follows trivially from Facts 3–5.

Proof of Claim 2: Let $n \in \omega$. We prove that $I_{n+1}(B)$ is canonically well-generated. For every Boolean algebra C, $I_1(C)$ and $I_2(C)$ are canonically well-generated, so we may assume that $n \geq 2$.

For every $a \in A$, $I(a)$ is countable, and hence by Proposition 2.8(d), it is canonically well-generated. For every $a \in A$ let $H(a) \subseteq \widehat{\mathrm{At}}(I(a))$ be an ESR for $I(a)$. The existence of such a set is assured by Proposition 2.8(b)(ii).

Let $a \in A$. If $\mathrm{rk}(d(a)) < n+1$ let $H_1(a) = \{d(a))\}$, otherwise let $H_1(a) = \emptyset$. Define $H^2_n(a) \subseteq I^2(a)$ to be the union of the following sets:

$$D_1(a) = \{\{e\} \mid e \in a \times \omega\},$$
$$D_2(a) = \{\{i\} \times (\omega - \{0, \ldots, n-1\}) \mid i \in a\},$$
$$D_3(a) = \{(b \times \omega) - (v(a) \times \{0, \ldots, n-1\}) \mid b \in H(a), \mathrm{rk}^{I(a)}(b) < n, \text{ and } b \text{ is not a singleton}\},$$
$$D_4(a) = H_1(a).$$

Note that in $D_3(a)$ we have that $(b \times \omega) \cap (v(a) \times \{n\})$ is finite. This follows from Clause (4) in the definition of $B(a)$, which says that $b \cap v(a)$ is finite. So we conclude that $(b \times \omega) - (v(a) \times \{n\}) \sim b \times \omega$.

Let $H^2_n = \bigcup_{a \in A} H^2_n(a)$. It follows from Facts 3 and 4 and from the fact that $H(a) \subseteq \widehat{\mathrm{At}}(I(a))$, that $H^2_n \subseteq \widehat{\mathrm{At}}(I(B)) \cap I_{n+1}(B)$. Facts 3, 4 and 7 also imply that for every $b \in \widehat{\mathrm{At}}(I(B)) \cap I_{n+1}(B)$, $|H^2_n \cap (b/{\sim})| = 1$. Hence H^2_n is a CSR for $I_{n+1}(B)$.

We show that if $b, c \in H^2_n$ and $b \preceq c$, then $b \leq c$. If $\mathrm{rk}(b) = 0$, then there is nothing to prove. So we assume that $\mathrm{rk}(b) \geq 1$. A direct computation shows that if there is $a \in A$ such that $b, c \in H^2_n(a)$, then $b \leq c$. Suppose that $b \in H^2_n(a)$, $c \in H^2_n(a')$ and $a \neq a'$. Note that b and c intersect every set of the form $\{\alpha\} \times \omega$ in a finite or a cofinite set. Since $a \cap a'$ is finite, $b \subseteq a \times \omega$ and $c \subseteq a' \times \omega$, it follows that there are finite sets $\sigma \subseteq a \cap a'$ and $\tau \subseteq \kappa \times \omega$ such that $b \cap c = (\sigma \times \omega) \triangle \tau$. So $\mathrm{rk}(b \cap c) \leq 1$. Since also $b \preceq c$, $\mathrm{rk}(b) = 1$. Any member of $H^2_n(a)$ whose rank is 1 belongs to $D_2(a)$. Hence b has the form $\{i\} \times (\omega - \{0, \ldots, n\})$. Since $b \preceq c$, $i \in a'$. So $b \leq c$.

We have shown that H^2_n is an ESR for $I_{n+1}(B)$. So $I_{n+1}(B)$ is canonically well-generated. This implies that $\mathrm{cl}^{\mathrm{bln}}(I_{n+1}(B))$ is canonically well-generated.

Proof of Claim 3: Suppose by contradiction that G is a well-founded lattice which generates B. Since B is unitary, $I(B)$ is a maximal ideal. By Proposition 2.8(c), we may assume that $G \subseteq I(B)$. Observe that for every $a \in I(B)$ there is $g \in G$ such that $a \leq g$. So $g_a := \min(\{g \in G \mid a \preceq g\})$ is well-defined. For every $\alpha \in \kappa$ and $\ell \in \omega$ let $c(\alpha) = \{\alpha\} \times \omega$ and

$$n(\alpha) = \min(\{n \in \omega \mid g_{c(\alpha)} \supseteq \{\alpha\} \times (\omega - \{0, \ldots, n-1\})\}).$$

Define $K_\ell = \{\alpha \in \kappa \mid n(\alpha) = \ell\}$. Then $\{K_\ell \mid \ell \in \omega\}$ is a partition of κ. So since $\langle \vec{A}, v\rangle$ is a spaced intersection system, there is $n_0 \in \omega$ such that $M := \mathrm{IS}(\vec{A}, v, K_{n_0})$ is infinite. Let $k \in M$ be such that $k \geq n_0$, and let $a \in A_k$ be such that $v(a) \cap K_{n_0}$ is infinite.

We show that for every $d \in I(B)$, $d \cap (v(a) \times \{k\})$ is finite. For every $a' \in A - \{a\}$, $a' \cap a$ is finite. Let $b \in I(a')$. Then

$$(b \times \omega) \cap (v(a) \times \{k\}) \subseteq (a' \times \omega) \cap (a \times \{k\}) = (a' \cap a) \times \{k\}.$$

So $(b \times \omega) \cap (v(a) \times \{k\})$ is finite. A similar computation shows that $d(a') \cap (a \times \{k\})$ is finite.

If $b \in I(a)$, then by Clause (4) in the definition of $B(a)$, $b \cap v(a)$ is finite. So $(b \times \{k\}) \cap (v(a) \times \{k\})$ is finite. By its definition $d(a) \cap (v(a) \times \{k\}) = \emptyset$.

The set $\bigcup\{b \times \omega \mid a' \in A,\ b \in I(a')\} \cup \{d(a') \mid a' \in A\} \cup \{\{e\} \mid e \in \kappa \times \omega\}$ generates $I(B)$ (as a ring), and we have shown that every member of this set intersects $v(a) \times \{k\}$ in a finite set. So for every $d \in I(B)$, $d \cap (v(a) \times \{k\})$ is finite.

Since $g_{d(a)} \in I(B)$, (†) $g_{d(a)} \cap (v(a) \times \{k\})$ is finite. Let

$$a' = \{\alpha \in v(a) \mid \{\alpha\} \times \omega \preceq g_{d(a)}\}.$$

Since $d(a) \preceq g_{d(a)}$ and $v(a) \subseteq a$, $v(a) - a'$ is finite. Hence, since $v(a) \cap K_{n_0}$ is infinite, also $a' \cap K_{n_0}$ is infinite. Let $\alpha \in a' \cap K_{n_0}$ be such that

$$\langle \alpha, k\rangle \notin g_{d(a)}. \tag{1}$$

(By (†), such an α exists.) Since $\alpha \in K_{n_0}$, it follows that

$$g_{c(\alpha)} \supseteq \{\alpha\} \times (\omega - \{0, \ldots, n_0 - 1\}).$$

Since $k \geq n_0$,

$$\langle \alpha, k\rangle \in g_{c(\alpha)}. \tag{2}$$

But then $g_{c(\alpha)} \not\leq g_{d(a)}$. Recall that $\{\alpha\}\times\omega = c(\alpha)$ and that $\alpha \in a'$. So

(3) $$c(\alpha) \preceq g_{d(a)}.$$

Facts 1–3 contradict the minimality of $g_{c(\alpha)}$. It follows that B is not well-generated. ∎

3. Two observations

As was explained in the paragraph following Proposition 1.2, a thin tall non-well-generated Boolean algebra gives rise to a canonical chain of length $\aleph_1$ of well-generated Boolean algebras whose union is not well-generated. We next prove the existence of such a Boolean algebra.

Definition 3.1: Let B be a superatomic BA. B is **thin tall**, if $\mathrm{rk}(B) = \aleph_1$, B is unitary, and for every $\alpha < \aleph_1$, $|I_\alpha(B)| = \aleph_0$. A topological space which is the Stone space of a thin tall Boolean algebra is called a **thin tall space**.

Note that a thin tall Boolean algebra is embeddable in $\mathcal{P}(\omega)$.

We shall use the following theorem of Dow and Simon [DS] Theorem 2.8.

THEOREM A: *Let $\mathcal{Y}$ be a countable set of locally compact locally countable spaces. Then there is a thin tall space X such that for every $Y \in \mathcal{Y}$, the one point compactification of Y is embeddable in X.*

THEOREM 3.2: *There is a thin tall non-well-generated Boolean algebra.*

Proof: Let X be a thin tall space such that the ordinal space $\aleph_1 + 1$ is embeddable in X. Let B be the clopen algebra of X. Then (i) B is embeddable in $\mathcal{P}(\omega)$, and (ii) the interval algebra of $\langle \aleph_1, < \rangle$ is a homomorphic image of B. Corollary 2.5 in [BR2] says that a Boolean algebra with properties (i) and (ii) above is not well-generated. So B is as required. ∎

The construction of a filling system described in Theorem 2.3 can be generalized to the class of cardinals $\{\beth_{\nu+1} \mid \mathrm{cf}(\nu) = \aleph_0\}$.

THEOREM 3.3: *Let ν be an ordinal with cofinality $\aleph_0$. Then $2^{\beth_\nu}$ has a $\beth_\nu^+$-deep filling system.*

Proof: For every infinite cardinal λ there is a complete metric space X with weight λ and cardinality $\lambda^{\aleph_0}$. For example, take Y to be the one-point compactification of a discrete space with cardinality λ and let $X = C(Y)$ be the set of all continuous functions from Y to $\mathbb{R}$ equipped with the uniform topology. Let $\lambda = \beth_\nu$. So $2^\lambda = \lambda^{\aleph_0}$. Let $\{e_\alpha \mid \alpha < 2^\lambda\}$ be an enumeration of all subsets e of X with cardinality λ and such that $|\operatorname{cl}^X(e)| = 2^\lambda$. We define by induction on $\alpha < 2^\lambda$ a sequence $\vec{A}_\alpha = \{A_{\alpha,i} \mid i \in \omega\}$. For every i, $A_{\alpha,i} \subseteq \mathcal{P}_{\aleph_0}(X)$. The induction hypotheses are:

(1) For every $i \in \omega$, $|A_{\alpha,i}| \leq |\alpha| + \aleph_0$.

(2) For every $i \neq j$, $A_{\alpha,i} \cap A_{\alpha,j} = \emptyset$ and $\bigcup_{i\in\omega} A_{\alpha,i}$ is an almost disjoint family. For every $a \in \bigcup_{i\in\omega} A_{\alpha,i}$, a is the range of a convergent sequence in X.

(3) If $\alpha < \beta$, then for every $i \in \omega$, $A_{\alpha,i} \subseteq A_{\beta,i}$.

Let $A_{0,i} = \emptyset$ for every $i \in \omega$. For a limit ordinal δ and $i \in \omega$ let $A_{\delta,i} = \bigcup_{\alpha<\delta} A_{\alpha,i}$. Suppose that $\vec{A}_\alpha$ has been defined, and we define $\vec{A}_{\alpha+1}$. Let $L_\alpha = \{\lim a \mid a \in \bigcup_{i\in\omega} A_{\alpha,i}\}$. So $|L_\alpha| \leq |\alpha| + \aleph_0$. Since $|\operatorname{cl}^X(e_\alpha)| = 2^\lambda$, the set $\operatorname{Acc}(e_\alpha)$ of accumulation points of e_α is of cardinality 2^λ, and thus $\operatorname{Acc}(e_\alpha) - L_\alpha \neq \emptyset$. Let $r_\alpha \in \operatorname{Acc}(e_\alpha) - L_\alpha$. Let $\{\vec{a}_{\alpha,i} \mid i \in \omega\}$ be a set of pairwise disjoint sequences converging to r_α such that for every $i \in \omega$, $a_{\alpha,i} := \operatorname{Rng}(\vec{a}_{\alpha,i}) \subseteq e_\alpha$. Let $A_{\alpha+1,i} = A_{\alpha,i} \cup \{a_{\alpha,i}\}$. It is easy to see that $\{A_{\alpha+1,i} \mid i \in \omega\}$ satisfies the induction hypotheses.

For every $i \in \omega$ let $A_i = \bigcup\{A_{\alpha,i} \mid \alpha < 2^\lambda\}$ and $\vec{A} = \{A_i \mid i \in \omega\}$. We show that $\vec{A}$ is a λ^+-deep filling system for X. Clearly, $\vec{A}$ is a candidate for X. Let $c \subseteq X$ and $|c| \geq \lambda^+$. Since the weight of c is $\leq \lambda$, there is $e \subseteq c$ such that $|e| = \lambda$ and e is dense in c. Denote the weight e by μ. Then $\mu^{\aleph_0} \geq |\operatorname{cl}^X(e)| > \lambda$. So $\mu^{\aleph_0} = \lambda^{\aleph_0}$.

A theorem of A. H. Stone ([KV] Chapter 1, Theorem 8.3) says that if Z is a complete metric space with weight μ and $|Z| > \mu$, then $|Z| = \mu^{\aleph_0}$.

It follows that $|\operatorname{cl}^X(e)| = \lambda^{\aleph_0}$. Hence for some $\alpha < 2^\lambda$, $e = e_\alpha$. Hence for every $i \in \omega$, $a_{\alpha,i} \subseteq e_\alpha \subseteq c$. So $\vec{A}$ is a filling system for X. ∎

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